Tangle diagrams

Let T be a tangle diagrams

- $P_T :=$ vector space generated by endpoints of T
- $V_T :=$ vector space generated by exterior faces of T
- $U_T :=$ vector space generated by interior faces of T
- $M_T :=$ quadratic form on $U_T \oplus V_T$ by [doing a thing at each crossing]

Let $\delta : V_T \hookrightarrow P_T$ be the map that embeds V_T into P_T by sending each face to its first endpoint (going counterclockwise).
Operations on tangles:

- **Disjoint union**: $T_1, T_2 \mapsto T_1 \sqcup T_2 = T_1 \sqcup_{p_1,p_2} T_2$
- **Capping**: $T \mapsto \widetilde{T} = \widetilde{T}_p$

3-tuples

Let P be a vector space. Let $\mathcal{T}(P)$ be 3-tuples $\{(n, W, B)\}$ where:

- $n \in \mathbb{Z}$
- $W \subseteq P$ a subspace
- $B : W \to W^*$ a quadratic form

Operations on 3-tuples:

- Adding: $(n_1, W_1, B_1) + (n_2, W_2, B_2) := (n_1 + n_2, W_1 \oplus W_2, B_1 \oplus B_2) \in \mathcal{T}(P_1 \oplus P_2)$.
- Pullback by $\phi : P_1 \to P_2$: $\phi^*(n_2, W_2, B_2) = (n_2, \phi^{-1}(W_2), \phi^* B_2 \phi)$

A map s from tangle diagrams to 3-tuples

Let s be a map that sends T to $s(T) = (n_T, W_T, B_T) \in \mathcal{T}(P_T)$ where:

- $n_T = \sigma(M|_{U_T})$
- $W_T = \{v \in V_T : M(v)|_{U_T} \in \operatorname{im}(M|_{U_T})\}$
- $B_T : W_T \to W_T^*$ is given by $B_T(v_1)(v_2) := M(v_1 - u_1)(v_2 - u_2)$ where $u_1, u_2 \in U_T$ are such that $M(u_1)|_{U_T} = M(v_1)|_{U_T}$ and $M(u_2)|_{U_T} = M(v_2)|_{U_T}$

Note that:

- B_T does not depend on the choices of u_1, u_2.
- Can decompose $U^* \cong \operatorname{im}(M|_U) \oplus \ker(M|_U)^*$ so can also describe W_T as:
  - $\{v \in V_T : M(v)|_{\ker(M|_U)} = 0\}$ or
  - the image of the projection of $\{w \in U_T \oplus V_T : M(w)|_U = 0\}$ to V_T.

How operations on tangles change s:

- For disjoint union: $s(T_1 \sqcup T_2) = s(T_1) \oplus s(T_2)$
- For capping:
  - If there is $v \in W_T$ that becomes an interior face in $\widetilde{T}$, then $s(\widetilde{T}) = i^*(n_T + B_T(v)(v), \operatorname{Ann}_{B_T}(v), B_T|_{\operatorname{Ann}_{B_T}(v)})$
  - If there is no such v, then $s(\widetilde{T}_p) = i^*(s(T))$

  where $i : P_{\widetilde{T}} \to P_T$ is the identity everywhere except $p_a \mapsto p_a + p_b$ where p_a, p_b are the endpoints before and after p.

Theorem 0.1. *If $s(T_1) = s(T_2)$, then:*

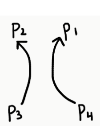
- *$s(T_1 \sqcup T) = s(T_2 \sqcup T)$ for any T*
- *$s(\widetilde{T_1}) = s(\widetilde{T_2})$*

A tangle invariant

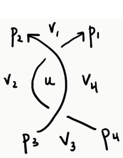
Theorem 0.2. *s is a tangle invariant.*

Proof. By the previous theorem, just need to show that applying s to the two sides of each Reidemeister move gives the same thing. ∎

R2:



$n_T = 0$
$W_T = \langle p_1 + p_3, p_2, p_4 \rangle$
$B_T = 0$



	u	v_1	v_2	v_3	v_4
u	0	-1	0	1	0
v_1	-1	$-v$	$-u$	u	$-u$
v_2	0	$-u$	0	u	0
v_3	1	u	u	v	u
v_4	0	$-u$	0	u	0

$n_T = 0$
$W_T = \langle v_1 + v_3, v_2, v_4 \rangle$
$= \langle p_1 + p_3, v_2, v_4 \rangle$
$B_T = 0$